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On the Velo–Zwanziger phenomenon

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Abstract

The Rarita–Schwinger equation in a curved background and an external electromagnetic field is discussed. We analyse the equation in the 2-component spinor formalism and derive consistency conditions for them. We derive a system of hyperbolic evolution equations together with constraints which is equivalent to the Rarita–Schwinger equation. We show that the constraints are not satisfied everywhere unless the consistency conditions are satisfied. These in turn require the background to be an Einstein manifold and the electromagnetic field to vanish.

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1. Introduction

The Rarita–Schwinger equation [1] has a number of peculiar properties. In the massless case the equation can be regarded as one of Dirac's relativistic wave equations [2] for particles with spin $3/2$. Fierz [3, 4] had pointed out that there exist hierarchies of such equations which describe the same one-particle states. One can move within the hierarchy by taking appropriate derivatives. In this sense the massless Rarita–Schwinger field is related to the usual zero rest mass field equation for spin $3/2$ [5] by one derivative. Fierz also observed that in general the solutions of these equations are not unique but only defined up to 'gauge solutions' which do not contribute to the energy and angular-momentum expressions constructed from the fields. This is the case also for the massless Rarita–Schwinger field.

A solution of the massless Rarita–Schwinger equation gives rise to such a 'potential modulo gauge' description of a spin $3/2$ field in close analogy to the Maxwell (i.e. the spin 1) case, where the electromagnetic field can be obtained from a potential which itself satisfies but is not completely determined by a field equation. There is still a gauge freedom present which must be fixed before the potential can be uniquely determined by its field equation.

While the (direct) formulation for a spin 3/2 field in terms of the conventional zero rest mass equation becomes inconsistent in any conformally curved spacetime (see, e.g., [5]) this is not so for the 'potential modulo gauge' description. Here again there are consistency conditions to be satisfied but in this case they involve only the Ricci tensor. This is a remarkable fact because the Rarita–Schwinger equation seems to be the only system of spinor equations where only the Ricci tensor appears in the obstruction to consistency. Thus, the vacuum Einstein equations can be considered as being 'integrability conditions' for the massless Rarita–Schwinger equation. This observation has been one motivation for an attempt to reconcile twistor theory with arbitrary (vacuum) spacetimes [6]. It also plays a fundamental role in the theory of super-gravity.

The massive case has been studied by various authors, in particular, by Velo and Zwanziger [7] who discuss the massive Rarita–Schwinger field coupled to an external electromagnetic field and by Madore [8] who in addition coupled the field to a linearized gravitational field. Their result is that in these circumstances the Rarita–Schwinger field seems to propagate acausally in the sense that the characteristics of the equation become space-like so that information about the field configuration can travel at speeds larger than the speed of light. This effect has been termed the 'Velo–Zwanziger phenomenon'.

In this paper we offer another analysis of the massive Rarita–Schwinger equations on a curved background in an exterior electromagnetic field. The plan of the paper is as follows. We start in section 2 by first translating the equations into the 2-component spinor formalism decomposing fields and equations into irreducible parts. The purpose of this somewhat lengthy exercise is to separate cleanly the various parts of the field in order to track their propagation properties individually.

Next, in section 3 we derive Buchdahl conditions for solutions of the Rarita–Schwinger equation. These are relations which necessarily hold between the solutions and the external fields such as the curvature of the background manifold or an external electromagnetic field. It will turn out that the Buchdahl conditions require that for consistency the manifold has to be an Einstein manifold and that there cannot be an external electromagnetic field. This is in contrast to earlier results because it implies that there is no Velo–Zwanziger phenomenon because one simply cannot couple the field consistently to an electromagnetic field in the first place.

Then, in section 4 we derive the 3 + 1 decomposition of these equations in the special case of flat Minkowski spacetime and vanishing electromagnetic field using the space-spinor formalism. We derive a system of symmetric hyperbolic evolution equations together with constraints. We find that the evolution system is underdetermined in the sense that there is a part of the field which is not determined by the evolution. Since this part is also not present in the constraints this implies that it can be specified freely which we take as an indication that the Rarita–Schwinger equation does not determine a unique solution. This arbitrariness can be interpreted in the case of massless fields as a gauge transformation of the kind discussed by Fierz. In the massive case, this is not possible.

We also discuss, in section 5, the propagation equations for the constraints. It will turn out that, in the massive case, the constraints do not propagate even in flat space without external electromagnetic fields unless the arbitrariness in the fields is fixed which amounts to switching to the Dirac formulation of the spin 3/2 field. Since the condition which arises is exactly the one coming from the Buchdahl condition for the special case of flat space but non-vanishing electromagnetic field we argue, that in the general case the constraints will propagate only, if the Buchdahl condition is satisfied. The propagation of the constraints is necessary for the well posedness of the Rarita–Schwinger system because only if we have this property are we able to conclude that a solution of the evolution equations evolved from initial values which satisfy the constraints will satisfy the constraints also at any later time. If this is not the case, then we do not have a solution of the entire Rarita–Schwinger system and hence there is no well posedness.

We conclude the paper with a short summary of our conclusions.

2. Translation to 2-component spinors

The Rarita–Schwinger equation [1] in an external electromagnetic field was formulated in [7] in terms of Dirac spinors:

$$(\mathrm{i}\,\Gamma \cdot \nabla - B)_a{}^d \psi_d = 0. \tag{1}$$

Here, ψ_d is a 1-form taking values in a 'charged bundle' of Dirac spinors over spacetime $\mathcal{M}$. This is the usual Dirac bundle equipped with an action of the group $U(1)$. Correspondingly, the spacetime connection, denoted by D_a, is promoted to a 'charged connection' ∇_a by 'minimal coupling', $\nabla_a = D_a - \mathrm{i}\,eA_a$. In (1) the differential operator is given by the term

$$\mathrm{i}\,\Gamma \cdot \nabla = \mathrm{i}\,\gamma^5 \varepsilon_{ab}{}^{cd} \nabla_c \gamma^b. \tag{2}$$

In order to derive and discuss the consistency conditions for this equation it is useful to translate it first into the formalism of 2-component spinors as presented in [5]. Note that we use these conventions throughout the paper.

We represent the Dirac spinor valued 1-form ψ_d as

$$\psi_d = \begin{pmatrix} \phi_{DD'S} \\ \chi_{DD'S'} \end{pmatrix}. \tag{3}$$

The Dirac matrices are represented in the form given in [5] as[1]

$$\gamma_a = \sqrt{2}\begin{pmatrix} 0 & \varepsilon_{AR}\varepsilon_{A'}{}^{S'} \\ \varepsilon_{A'R'}\varepsilon_A{}^S & 0 \end{pmatrix} \qquad \gamma^5 = \begin{pmatrix} -\mathrm{i}\,\varepsilon_R{}^S & 0 \\ 0 & \mathrm{i}\,\varepsilon_{R'}{}^{S'} \end{pmatrix}. \tag{4}$$

The mass term in (1) is given by $B = B_a{}^b = m\gamma_a{}^b$ where γ_{ab} are the matrices

$$\gamma_{ab} = \tfrac{1}{2}(\gamma_a\gamma_b - \gamma_b\gamma_a). \tag{5}$$

Represented in terms of 2-component spinors these matrices read

$$\gamma_{ab} = 2\begin{pmatrix} \varepsilon_{A'B'}\varepsilon_{R(A}\varepsilon_{B)}{}^S & 0 \\ 0 & \varepsilon_{AB}\varepsilon_{R'(A'}\varepsilon_{B')}{}^{S'} \end{pmatrix}. \tag{6}$$

Inserting these representations into (1) we obtain after some calculation the system of equations

$$\begin{aligned} \nabla_{AB'}\chi_{BA'}{}^{B'} - \nabla_{BA'}\chi_{AB'}{}^{B'} &= m\phi_{A'AB} \\ \nabla_{BA'}\phi_{B'A}{}^{B} - \nabla_{AB'}\phi_{A'B}{}^{B} &= m\chi_{AA'B'}. \end{aligned}$$

Note, that in the case $m = 0$ the two equations decouple. If we assume for the moment that the spacetime is Minkowski space and that there is no electromagnetic field present then it is obvious that there is an arbitrariness in these equations. We are free to replace $\chi_{AA'B'}$ and $\phi_{AA'B}$ with $\chi_{AA'B'} + \nabla_{AA'}\chi_{B'}$ and $\phi_{AA'B} + \nabla_{AA'}\phi_B$ for arbitrary spinor fields $\chi_{B'}$ and ϕ_B without changing the equations. Thus, whenever $\chi_{AA'B'}$ and $\phi_{AA'B}$ are solutions then so are $\chi_{AA'B'} + \nabla_{AA'}\chi_{B'}$ and $\phi_{AA'B} + \nabla_{AA'}\phi_B$. This is a well-known property of the massless Rarita–Schwinger equations [2, 3, 6, 9]. It implies that a solution of the equations for $m = 0$ can

[1] Note that the Clifford relation obeyed by the Dirac matrices used in [7] and [5] differ by a sign. This is compensated for in the formulae.

be determined only *up to gauge transformations* of the above form and the equations can be regarded as determining a spin 3/2 field in a 'potential modulo gauge' description.

In the case $m \neq 0$ the same replacement does not yield a new solution because now the condition that the changed fields be again a solution implies that $\chi_{B'}$ has to be covariantly constant. And this, in turn, implies that the fields remain in fact unchanged. So in this case there is no notion of a field being given by a potential modulo gauge description.

The final step in rewriting the Rarita–Schwinger equation is to decompose the fields and the equations into irreducible parts. Thus, we write

$$\phi_{AA'B} = \sigma_{A'AB} + \varepsilon_{AB}\sigma_{A'}$$
$$\chi_{AA'B'} = \tau_{AA'B'} + \varepsilon_{A'B'}\tau_A$$

where now the fields $\sigma_{AA'B}$ and $\tau_{AA'B'}$ are symmetric in their last pair of indices. Then we obtain the following system of spinor equations:

$$\nabla_{B'(A}\tau_{B)A'}{}^{B'} - \nabla_{A'(A}\tau_{B)} = m\sigma_{A'AB} \tag{7}$$

$$\nabla_{BB'}\tau^{BB'}{}_{A'} + 3\nabla_{BA'}\tau^{B} = -6\,m\sigma_{A'} \tag{8}$$

$$\nabla_{B(A'}\sigma_{B')A}{}^{B} - \nabla_{A(A'}\sigma_{B')} = m\tau_{AA'B'} \tag{9}$$

$$\nabla_{BB'}\sigma^{B'B}{}_{A} + 3\nabla_{AB'}\sigma^{B'} = -6\,m\tau_{A}. \tag{10}$$

This is the set of equations we will analyse in the following sections.

3. Consistency conditions

Before we analyse the Rarita–Schwinger system in terms of hyperbolic evolution equations and constraints we derive the so-called Buchdahl conditions. These relations between fields and external quantities arise when we try to impose a system of equations on an arbitrarily curved manifold and for a given exterior electromagnetic field. To this end we take further covariant derivatives of the equations, commute them in order to introduce curvature terms and try to eliminate all derivatives of the fields. If this is possible we will end up with an algebraic relation between the fields, the curvature and the exterior electromagnetic field.

For later convenience we introduce the notation

$$[\nabla_{AA'}, \nabla_{BB'}] = \varepsilon_{AB}\Box_{A'B'} + \varepsilon_{A'B'}\Box_{AB} \tag{11}$$

where the curvature derivations $\Box_{AB}$ and $\Box_{A'B'}$ here also contain the electromagnetic field $F_{ab} = \varepsilon_{AB}F_{A'B'} + \varepsilon_{A'B'}F_{AB}$. Explicitly, we have for any spinor κ_C (see [5])

$$\Box_{AB}\kappa_C = -\Psi_{ABC}{}^{D}\kappa_D + 2\Lambda\varepsilon_{C(A}\kappa_{B)} + \mathrm{i}\,eF_{AB}\kappa_C$$
$$\Box_{A'B'}\kappa_C = -\Phi_{A'B'C}{}^{E}\kappa_E - \mathrm{i}\,eF_{A'B'}\kappa_C.$$

We first take a derivative of (7).

$$\begin{aligned} m\nabla_C{}^{A'}\sigma_{A'BA} &= \nabla_C{}^{A'}\nabla_{B'(A}\tau_{B)A'}{}^{B'} - \nabla_C{}^{A'}\nabla_{A'(A}\tau_{B)} \\ &= -\varepsilon_{C(A}\Box^{A'B'}\tau_{B)A'B'} + \nabla_{B'(A}\nabla_C{}^{A'}\tau_{B)A'}{}^{B'} + \nabla_{CA'}\nabla^{A'}{}_{(A}\tau_{B)} \\ &= -\varepsilon_{C(A}\Box^{A'B'}\tau_{B)A'B'} + \tfrac{1}{2}\nabla_{AB'}\left[\nabla_C{}^{A'}\tau_{BA'}{}^{B'}\right] \\ &\quad + \tfrac{1}{2}\nabla_{BB'}\left[\nabla_C{}^{A'}\tau_{AA'}{}^{B'}\right] + \tfrac{1}{2}\varepsilon_{C(A}\Box\tau_{B)} + \Box_{C(B}\tau_{A)}. \end{aligned}$$

Now we use (7) and (8) to obtain

$$m\nabla_C{}^{A'}\sigma_{A'BA} = -\varepsilon_{C(A}\Box^{A'B'}\tau_{B)A'B'} + \tfrac{1}{2}\varepsilon_{C(A}\Box\tau_{B)} + \Box_{C(B}\tau_{A)}$$
$$+ \tfrac{1}{2}\nabla_{AB'}\big[-\nabla^{B'}{}_{(C}\tau_{B)} - m\sigma^{B'}{}_{BC} - 3\varepsilon_{BC}\big(m\sigma^{B'} + \nabla_D{}^{B'}\tau^D\big)\big]$$
$$+ \tfrac{1}{2}\nabla_{BB'}\big[-\nabla^{B'}{}_{(C}\tau_{A)} - m\sigma^{B'}{}_{AC} - 3\varepsilon_{AC}\big(m\sigma^{B'} + \nabla_D{}^{B'}\tau^D\big)\big]$$
$$= -\varepsilon_{C(A}\Box^{A'B'}\tau_{B)A'B'} + \tfrac{1}{2}\varepsilon_{C(A}\Box\tau_{B)} + \Box_{C(B}\tau_{A)} - \tfrac{1}{4}\varepsilon_{A(C}\Box\tau_{B)} - \tfrac{1}{2}\Box_{A(C}\tau_{B)}$$
$$- \tfrac{1}{2}m\nabla_{AB'}\sigma^{B'}{}_{BC} - \tfrac{3}{2}\varepsilon_{BC}\big(m\nabla_{AB'}\sigma^{B'} + \nabla_{B'A}\nabla^{B'}{}_D\tau^D\big) - \tfrac{1}{4}\varepsilon_{B(C}\Box\tau_{A)}$$
$$- \tfrac{1}{2}\Box_{B(C}\tau_{A)} - \tfrac{1}{2}m\nabla_{BB'}\sigma^{B'}{}_{AC} - \tfrac{3}{2}\varepsilon_{AC}\big(m\nabla_{BB'}\sigma^{B'} + \nabla_{BB'}\nabla^{B'}{}_D\tau^D\big).$$

Collecting appropriate terms we get

$$m\nabla_C{}^{A'}\sigma_{A'BA} + m\nabla_{B'(B}\sigma^{B'}{}_{A)C} - 3m\varepsilon_{C(A}\nabla_{B)B'}\sigma^{B'}$$
$$= -\varepsilon_{C(A}\Box^{A'B'}\tau_{B)A'B'} + \tfrac{3}{4}\varepsilon_{C(A}\Box\tau_{B)} - \tfrac{1}{2}\Box_{AB}\tau_C$$
$$+ \tfrac{1}{2}\Box_{C(A}\tau_{B)} + \tfrac{3}{2}\varepsilon_{C(A}\nabla_{B)B'}\nabla^{B'}{}_D\tau^D$$
$$= -\varepsilon_{C(A}\Box^{A'B'}\tau_{B)A'B'} + \tfrac{3}{4}\varepsilon_{C(A}\Box\tau_{B)} - \tfrac{1}{2}\Box_{AB}\tau_C - \tfrac{3}{4}\varepsilon_{C(A}\Box\tau_{B)} + \tfrac{3}{2}\varepsilon_{C(A}\Box_{B)D}\tau^D$$
$$= -\varepsilon_{C(A}\Box^{A'B'}\tau_{B)A'B'} - 2\Box_{AB}\tau_C + 2\Box_{C(A}\tau_{B)}.$$

Rewriting the left-hand side yields

$$-m\varepsilon_{C(B}\big(\nabla^{B'D}\sigma_{B'A)D} + 3\nabla_{A)B'}\sigma^{B'}\big) = -\varepsilon_{C(A}\Box^{A'B'}\tau_{B)A'B'} - 2\Box_{AB}\tau_C + 2\Box_{C(A}\tau_{B)}. \tag{12}$$

Using (10) we get

$$6m^2\varepsilon_{C(B}\tau_{A)} = -\varepsilon_{C(A}\Box^{A'B'}\tau_{B)A'B'} - 2\Box_{AB}\tau_C + 2\Box_{C(A}\tau_{B)}. \tag{13}$$

This expression is equivalent to its contraction so we finally obtain

$$H_A \equiv \Box^{A'B'}\tau_{AA'B'} - 2\Box_{AB}\tau^B + 6m^2\tau_A = 0. \tag{14}$$

We could also have taken a derivative of (8) and gone through the same procedure. Then we would have ended up with exactly the same relation as above. Introducing now the explicit form of the curvature derivations we obtain the consistency conditions

$$H_A = -\Phi_A{}^{BA'B'}\tau_{BA'B'} + 6\Lambda\tau_A - \mathrm{i}\,eF^{A'B'}\tau_{AA'B'} + 2\mathrm{i}\,eF_{AB}\tau^B + 6m^2\tau_A = 0. \tag{15}$$

This relation couples the values of the fields τ_A and $\tau_{AA'B'}$ with the curvature and the extrinsic electromagnetic field if there should be one. A similar relation holds for the fields $\sigma_{A'}$ and $\sigma_{AA'B}$. This relation can be obtained formally by complex conjugating (15) and then replacing $\bar{\tau}_{A'AB}$ and $\bar{\tau}_{A'}$ by $\sigma_{AA'B}$ and $\sigma_{A'}$, respectively. This yields

$$K_{A'} \equiv \Box^{AB}\sigma_{AA'B} - 2\Box_{A'B'}\sigma^{B'} + 6m^2\sigma_{A'} = 0. \tag{16}$$

It is not immediately obvious that such relations should exist because this depends very much on the detailed structure of the equation in question.

Let us now discuss relation (15) for various special cases. Suppose we are in Minkowski space and suppose that (7)–(10) hold. Since all the commutators vanish we necessarily recover the condition

$$6m^2\tau_B = 0. \tag{17}$$

Thus, either we have $m^2 = 0$ or $\tau_B = 0$. So if we insist on a massive field then this field cannot have a component τ_B and the spinor field $\chi_{BB'A'}$ must be symmetric in its last pair of indices. On the other hand, in the massless case, we may admit the part τ_B but it does not play a role. Instead it corresponds to the gauge freedom discussed in section 2.

Focussing now on a massive field in a general spacetime with an electromagnetic field we require $\tau_B = 0$ because this is necessary even in the flat case. Now (15) reduces to

$$0 = \Phi_A{}^{BA'B'}\tau_{BA'B'} + \mathrm{i}\, eF^{A'B'}\tau_{AA'B'}$$

which has to hold for all values of the field $\tau_{BA'B'}$. This is a severe algebraic restriction on the field and/or the electromagnetic and curvature fields. At any point in spacetime we can arrange for the field $\tau_{BA'B'}$ to take arbitrary values which implies that at any point we have $\Phi_{ABA'B'} + \mathrm{i}\, eF_{A'B'}\varepsilon_{AB} = 0$ which in turn implies that

(i) the spacetime is an Einstein manifold, $\Phi_{ABA'B'} = 0$,
(ii) the electromagnetic field vanishes, $F_{AB} = 0$.

While the first consequence is familiar from the massless case, the second consequence is new. It implies that it is impossible to couple a Rarita–Schwinger field to an electromagnetic field in a consistent way. This result is independent of the curvature of the spacetime, i.e. even in a flat background one cannot have an electromagnetic field present. Note, that the requirement that τ_B should vanish is not strictly necessary if the scalar curvature of the spacetime is a non-zero constant. Then it is possible to have $m^2 + \Lambda = 0$ with non-vanishing τ_B. However, this does not change the above conclusions.

4. 3 + 1 decomposition

In this section we will study the Rarita–Schwinger system from the point of view of the initial value problem. We want to derive a system of evolution equations and constraints which will be satisfied by any solution of the Rarita–Schwinger system and vice versa.

In order to find the basic propagation properties of this system of equations we need to perform a 3 + 1-splitting of the system. This is done as usual using the space-spinor formalism [10] (or [11] where this formalism has been applied in the massless case). To this end we fix a time-like vector field n^a, normalized by $n_a n^a = 2$, so that $n_{AA'}n^{BA'} = \varepsilon_A{}^B$. We use this vector field to 'convert the primed indices to unprimed ones'. Thus, for instance, we write the field $\tau_{AA'B'}$ as

$$n_C{}^{A'} n_B{}^{B'} \tau_{AA'B'} = t_{ABC} + 2\varepsilon_{A(C}t_{B)} \qquad (18)$$

where t_{ABC} is totally symmetric in all its indices. Similarly, the field $\sigma_{A'AB}$ yields two irreducible parts s_{ABC} and s_A and we set $\sigma_A = n_A{}^{A'}\sigma_{A'}$. The derivative operator $\nabla_{AA'}$ can be written in the form

$$\nabla_{AA'} = n_{AA'}\partial - n_{A'}{}^C\partial_{AC} \iff n_B{}^{A'}\nabla_{AA'} = \varepsilon_{AB}\partial + \partial_{AB}.$$

It is enough for our purposes to assume that the underlying spacetime is Minkowski space but we will allow the presence of an additional external electromagnetic field. Then we can arrange for the time-like vector field n^a to be covariantly constant. This has the consequence that all derivatives of the vector field vanish.

Inserting these decompositions into ((7)–(10)) yields six equations which we group into four evolution equations

$$\partial t_{ABC} + \partial_{(A}{}^D t_{BC)D} - \partial_{(AB}(t_{C)} - \tau_{C)}) = -m s_{ABC} \qquad (19)$$

$$\partial(t_A + \tau_A) - \tfrac{1}{3}\partial_{AB}(t^B + \tau^B) - \tfrac{2}{3}\partial_{AB}(t^B - \tau^B) = m(s_A - \sigma_A) \qquad (20)$$

$$\partial s_{ABC} - \partial_{(A}{}^D s_{BC)D} + \partial_{(AB}(s_{C)} - \sigma_{C)}) = m t_{ABC} \qquad (21)$$

$$\partial(s_A + \sigma_A) + \tfrac{1}{3}\partial_{AB}(s^B + \sigma^B) + \tfrac{2}{3}\partial_{AB}(s^B - \sigma^B) = -m(t_A - \tau_A) \qquad (22)$$

and two constraint equations

$$0 = T_A \equiv \partial^{BC} t_{BCA} + 2\partial_{AB}(t^B + \tau^B) + 3m(\sigma_A + s_A) \tag{23}$$

$$0 = S_A \equiv \partial^{BC} s_{BCA} + 2\partial_{AB}(s^B + \sigma^B) + 3m(t_A + \tau_A). \tag{24}$$

We note that in the case of a general curved background manifold we would have obtained equations with the same principal part but with additional lower order terms containing the derivatives of the time-like evolution vector n^a.

Now we can see the peculiar behaviour of this system:

- We only get propagation equations for the sums $t^A + \tau^A$ and $s^A + \sigma^A$ while the differences $t^A - \tau^A$ and $s^A - \sigma^A$ do not evolve. Since they are not present in the constraints either we have to conclude that they are not determined by the system. We are free to specify them arbitrarily during the course of the evolution. This suggests that these combinations are not physically meaningful quantities.
- The evolution equations, now regarded as equations for t_{ABC} and $t_A + \tau_A$, respectively for s_{ABC} and $s_A + \sigma_A$, are symmetric hyperbolic which is easily verified. The characteristics for equations (19) and (21) are the light cone and a time-like cone in the interior of the light cone, which is also the characteristic for equations (20) and (22). Therefore, for any arbitrary choice of the difference fields the Cauchy problem for the evolution equations is well posed. The propagation of the fields is causal in the sense that there is no superluminal propagation speed.

  However, if we do not regard the difference fields as being fixed but instead try to couple them to the propagating fields then we will change the characteristics. For example imposing the condition that the difference fields should be linearly dependent on the sum of the fields changes the characteristics in an almost arbitrary way. This should be taken as an additional hint that the difference of the fields is in a sense an unphysical feature of the equations.
- We have seen earlier from the consistency relations that the components τ_A and $\sigma_{A'}$ should vanish for a massive field. We could also require these fields to vanish here, thereby fixing the arbitrariness in the evolution. However, this will not change the conclusions. Furthermore, we consider this to be an artificial procedure because there is no intrinsic way to achieve this, e.g., by choosing appropriate initial conditions. Putting these fields to zero essentially amounts to switching from the Rarita–Schwinger formulation to the Dirac formulation of the spin 3/2 equation [2].
- In the case $m = 0$ the equations decouple into two sets consisting of two evolution equations and one constraint each. Let us consider the three equations (19), (20) and (23). Here again, the difference $t^A - \tau^A$ is not determined by the equations. The gauge transformation mentioned in section 2 now translates into the transformations

  $$t_{ABC} \mapsto t_{ABC} + \partial_{(AB}\chi_{C)}$$
  $$t_A \mapsto t_A - \tfrac{3}{2}\partial\chi_A + \tfrac{1}{2}\partial_{AB}\chi^B$$
  $$\tau_A \mapsto -\tau_A - \partial\chi_A + \partial_{AC}\chi^C.$$

  One can use this gauge transformation to make $\tau_A = 0$ (so that the original field $\chi_{AA'B'}$ is symmetric in its primed indices). This yields a system of two evolution equations and one constraint equation which has a well-posed Cauchy problem (see, e.g., [11]). In this way the indeterminacy in the system can be circumvented and one can still make sense of the equations. This argument can be generalized to Ricci flat spacetimes.

5. Propagation of constraints

The Rarita–Schwinger system is equivalent to the combined system of evolution and constraint equations derived above. Therefore, a solution of the Rarita–Schwinger system has to satisfy the constraint equations *at each instant of time*. We necessarily have to satisfy the constraints initially to provide the initial data for the evolution equations. But we need to verify that each solution of the constraints will be propagated by the evolution into another solution of the constraints. If this was not the case, then a solution of the evolution equations would violate the constraints even if it satisfied them initially.

The usual way to show the propagation of the constraints is to derive a subsidiary system of evolution equations for the constraints (i.e. for those fields whose vanishing amounts to the imposition of the constraints) which has the zero fields as a solution and for which uniqueness of solutions for given initial data holds. Then one can conclude that the constraints are propagated in the above sense.

We will now derive the system of equations satisfied by the fields T_A and S_A. Because these calculations are rather cumbersome we restrict ourselves to the special case already mentioned above where we assume the manifold to be flat but allow for an external electromagnetic field. This case is in a sense complementary to the one discussed in [11] where it was assumed that the manifold is curved but no electromagnetic field was admitted.

To do the calculation one needs to know the commutators between the derivative operators ∂ and ∂_{AB}. These are given by

$$[\partial, \partial_{AB}] = \tfrac{1}{2}(\hat{\Box}_{AB} - \Box_{AB})$$

$$\partial_{C(A}\partial_{B)}{}^{C} = \tfrac{1}{2}(\hat{\Box}_{AB} + \Box_{AB}).$$

Here we have defined $\hat{\Box}_{AB} = n_A{}^{A'} n_B{}^{B'} \Box_{A'B'}$, the space-spinor equivalent of the primed curvature derivation. In the present case, these derivations contain only the electromagnetic field strength. In the general case they would also contain terms of the form $K \cdot \partial$ where K stands for a derivative of n_a and ∂ is one of the derivative operators.

Taking a time derivative of T_A, commuting derivatives and using the evolution equations and the expressions for the commutators we arrive after some calculation at the equations

$$\partial T_A - \tfrac{1}{3}\partial_{AB} T^B + m S_A = \hat{\Box}^{BC} t_{ABC} + 2\hat{\Box}_{AB} t^B - 2\Box_{AB}\tau^B + 6m^2\tau_A$$

$$\partial S_A + \tfrac{1}{3}\partial_{AB} S^B + m T_A = \Box^{BC} s_{ABC} + 2\Box_{AB} s^B - 2\hat{\Box}_{AB}\sigma^B + 6m^2\sigma_A.$$

Consider now again expressions (14) and (16). Written in terms of space-spinors we have

$$H_A = \hat{\Box}^{CD} t_{ACD} + 2\hat{\Box}_{AB} t^B - 2\Box_{AB}\tau^B + 6m^2\tau_A \tag{25}$$

$$K_A = \Box^{CD} s_{ACD} + 2\Box_{AB} s^B - 2\hat{\Box}_{AB}\sigma^B + 6m^2\sigma_A. \tag{26}$$

So the propagation equations for the constraints have the form

$$\partial T_A - \tfrac{1}{3}\partial_{AB} T^B + m S_A = H_A \tag{27}$$

$$\partial S_A + \tfrac{1}{3}\partial_{AB} S^B - m T_A = -K_A. \tag{28}$$

We have obtained these equations with the additional assumption that the time-like vector n^a should be covariantly constant. However, the general equations would have had the same properties. In particular, the principal parts and the right-hand sides would have been the same. The only difference would have been some additional low order terms on the left-hand side involving the derivatives of n^a multiplied by S_A or T_A (see [11] for these equations in the massless case on a curved background).

6. Discussion and conclusion

The system (27) and (28) of propagation equations is symmetric hyperbolic. For vanishing right-hand sides it is also homogeneous. In that case we can conclude by standard arguments that the unique solution for vanishing initial data is the zero solution. This means that if we start with data for the evolution equations (19)–(22) which satisfy the constraints, i.e. for which $T_A = 0$ and $S_A = 0$, then we will obtain $T_A = 0 = S_A$ everywhere, i.e. the constraints will be satisfied at all times.

However, the obstruction to this conclusion is exactly the consistency conditions $H_A = 0$ and $K_A = 0$. Only when these are satisfied do we get the vanishing of the constraints everywhere. This implies that *only in that case* the Rarita–Schwinger system has a well-posed Cauchy problem in the sense of a hyperbolic system of evolution equations together with propagating constraints, i.e. only when the consistency conditions are satisfied. Therefore, only in that case there exists a solution of the Rarita–Schwinger system.

However, the consistency conditions require the background manifold to be an Einstein manifold and the external electromagnetic field to vanish. So we arrive at the conclusion that a spin $3/2$ field cannot be coupled consistently to an external electromagnetic field because this would violate the Buchdahl conditions. It can live on an Einstein spacetime for which the curvature scalar is related to the mass of the field by $\Lambda = -m^2$. In particular, a massless field can only exist on a vacuum spacetime.

How can we understand that there is no consistent coupling of the Rarita–Schwinger field to the electro-magnetic spin 1 field even though it can be coupled consistently to the gravitational field, i.e. the Weyl curvature which has spin 2? One would have expected that problems come with higher spin. Clearly, the explanation has to be the algebraic structure of the field. The fact, that the Rarita–Schwinger field contains parts which have mixed index type but at most two indices of the same kind allows the Weyl spinor to 'slip through' so that it does not appear in the consistency conditions (this would be different if there were more than two indices of one kind). All the other curvature and electromagnetic fields are coupled to the Rarita–Schwinger field in the consistency conditions and, therefore, are restricted. So the question is not whether we have low spin or high spin, but depends crucially on exactly which representations are present in the spinor field.

The earlier results in [8] and [7] suggest that the Rarita–Schwinger equation has solutions when coupled to an external field but that there are modes which propagate acausally. They seem to have been obtained by focussing on the evolution equations and fixing the relationship of the free difference fields in a certain way which changes the characteristics. However, it seems to have been ignored to check for the propagation of the constraints. The present result implies that there is no 'Velo–Zwanziger phenomenon' because a Rarita–Schwinger field cannot exist in an external electromagnetic field.

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References

[1] Rarita W and Schwinger J 1941 On a theory of particles with half-integral spin *Phys. Rev.* **60** 61
[2] Dirac P A M 1936 Relativistic wave equations *Proc. R. Soc.* A **155** 447–59

[3] Fierz M 1939 Über die relativistische Theorie kräftefreier Teilchen mit beliebigem Spin *Helv. Phys. Acta* **12** 3–37

[4] Fierz M 1940 Über den Drehimpuls von Teilchen mit Ruhemasse Null und beliebigem Spin *Helv. Phys. Acta* **13** 45–60

[5] Penrose R and Rindler W 1984 *Spinors and Spacetime* vol 1 (Cambridge: Cambridge University Press)

[6] Penrose R 1994 Twistors and the Einstein equations *Proc. Plymouth Conf.* ed S Huggett (New York: Marcel Dekker)

[7] Velo G and Zwanziger D 1969 Propagation and quantization of Rarita–Schwinger waves in an external electromagnetic potential *Phys. Rev.* **186** 1337–41

[8] Madore J 1975 The characteristic surfaces of a classical spin (3/2) field in an Einstein-Maxwell background *Phys. Lett.* B **55** 217–8

[9] Fierz M and Pauli W 1939 On relativistic wave equations for particles of arbitrary spin in an electromagnetic field *Proc. R. Soc.* A **173** 211–32

[10] Sommers P 1980 Space spinors *J. Math. Phys.* **21** 2567–71

[11] Frauendiener J 1995 On spin-(3/2) systems in Ricci flat space-times *J. Math. Phys.* **36** 3012–22